Stat 343 More Practice with Monte Carlo Integration

For all of the problems below, first write your answer as a suitable integral, then write a 2-step algorithm to approximate the integral.

1. Suppose $X \sim$ Normal$(5, 2^2)$. You have a function that can generate a sample $x_1, \ldots, x_m$ from this normal distribution. How could you use Monte Carlo integration to approximate $E(X^2)$?

2. Suppose you did the M&M's experiment and your posterior distribution is $\Theta | X_1, \ldots, X_n \sim$ Beta$(12, 136)$. You have a function that can generate a sample $\theta_1, \ldots, \theta_m$ from this beta distribution. How could you use Monte Carlo integration to approximate the posterior probability that Θ is between 0.1 and 0.2?

3. Suppose you have a normal model: $X_1, \ldots, X_n | \Theta, \Xi \overset{\text{i.i.d.}}{\sim} \text{Normal}(\theta, \xi^{-1})$. Neither θ nor ξ are known, so you put a prior distribution on them. As we have seen, there is no conjugate prior for both θ and ξ in this model, so you don't know an exact parametric distribution for the joint posterior of Θ and Ξ. However, you have a function that can generate a sample $(\theta_1, \xi_1), \ldots, (\theta_m, \xi_m)$ from the joint posterior of $\Theta, \Xi | X_1, \ldots, X_n$. How could you use Monte Carlo integration to approximate the posterior probability that Θ is between 1 and 2?

4. What theorem justifies all of the above? What do your answers to problems 2 and 3 have to do with an expected value?